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The Theory of Uniform Cables—Part II: Calculation of Charge Components

By T. A. LENAHAN

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The analysis of electromagnetic propagation over uniform cables depends on the calculation of the charge densities on the conductors, relative to a potential function that is not necessarily constant on the conductors. By considering such a potential function as the real part of an analytic function, two Laurent series are derived, one of which involves the Fourier components of the potential function and its associated charge densities on the conductors. The second series accounts for the relative location of the conductors. The two series are equated to give a system of linear equations that can be solved for the charge components. The results obtained, which apply to uniform cables whose conductors (including the shield, if present) have circular cross sections and are covered with two layers of dielectric insulation, can be used to calculate the propagation modes and propagation constants of the cable.

I. INTRODUCTION

Multiconductor cables for telecommunications have distributions of charge on each conductor. The surface charge density on the conductors is proportional to the normal derivative of a potential function (i.e., a solution to Laplace's equation), which is defined in the region separating the conductors and is constant on the conductors. The constant values of the potential are the voltages of the conductors, and the proportionality constant is the permittivity of the material next to the conductor. Also, generalized charge densities are defined for potential functions, such as the longitudinal component of the electric field, which are not constant on the conductors.

In the present work an algorithm is developed for computing the charge densities in this generalized sense for uniform cables. Thus, the conductors of the cable are assumed to be straight and parallel, so that each transverse cross section is identical. The wires are assumed to have

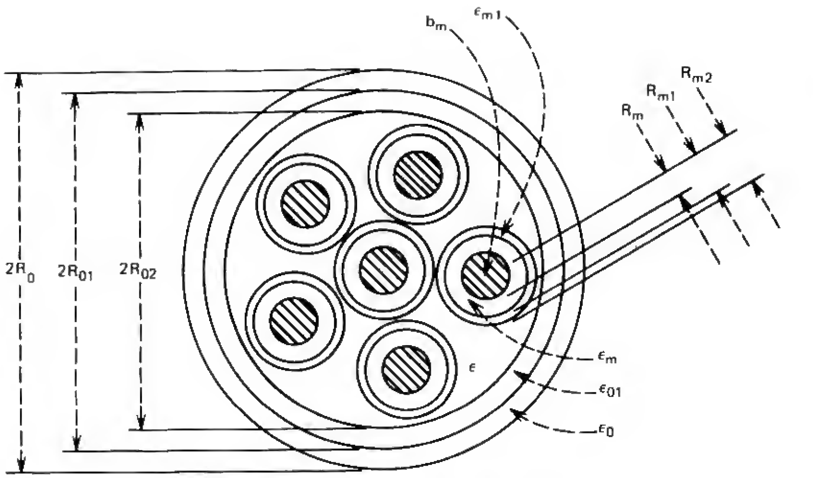

Fig. 1—Typical cable cross section

circular cross sections and to be covered by two circularly symmetric layers of homogeneous dielectric material. Surrounding the collection of wires is a circular, metallic shield, and it too is assumed to have two uniform layers of dielectric on its inside surface. A typical cross section is shown in Fig. 1.

Interest in the charge densities was spurred by the recent finding that the modes of a cable and their associated propagation constants can be expressed directly in terms of the charge densities when, as in Kuznetsov's work,[1] low frequencies are excluded. Subsequently, it was found that the analysis could be extended to low frequencies by using the notion of generalized charge densities associated with the longitudinal component of the electric field. These matters are developed in detail in Ref. 2.

Previous work on calculating charge on the conductors of a cable usually involved the simplifying assumption of homogeneous dielectric in the region separating the conductors. For two identical wires in free space, an explicit formula is available for all the Fourier components of the charge density on a wire (see Ref. 1, page 41). Goluzin,[3] using the theory of complex variables, has developed a continued fraction expansion for a potential function in a region bounded by circles; the expansion converges under certain conditions on the size and location of the circles. Nordgard[4] has developed an algorithm involving a matrix inversion for computing Fourier components of the charge densities for a pair in a shield. Also, the capacitance elements (the zero-order component of a charge density) have been calculated in a variety of circumstances.[5,6]

For inhomogeneous dielectrics, the calculation of charge density has

been approached by a finite-difference technique[7] coupled with a Fourier analysis[8] of the normal derivative to give the Fourier components. This technique, by calculating more information than is needed, has proved to involve more computer expense than the technique to be described here.

In this paper, techniques from the theory of complex variables are used to develop a set of linear equations for the charge components. Following Goluzin (see Ref. 3), the potential function is viewed as the real part of an analytic function; then two Laurent series are calculated for it about each conductor. The first is expressed in terms of the Fourier components of the charge density and the potential function on each conductor; the second is expressed in terms of singularities located at the centers of the conductors (at infinity for the shield). Since the two Laurent series must be identical, their corresponding coefficients can be equated to give a system of linear equations from which the charge components can be calculated. For convenience, the system of equations is expressed in matrix form.

The matrix equations are developed in the next section, with details of the calculations and proofs relegated to appendices. In Section III, the development is summarized and then extended to apply to cables without shields, to cables with holes in the dielectric, and to more general boundary problems that arise in determining the admittance matrix and other propagation parameters of the cable. Numerical examples and experimental testing of this work is presented in Ref. 9.

II. CALCULATION OF THE CHARGE COMPONENTS

The cable to be considered consists of M straight and parallel wires with circular cross sections surrounded by a circular shield of inside radius R_0. The wires and the inside surface of the shield are covered by two different layers of dielectric insulation. For the mth wire ($1 \leqq m \leqq M$) the radius is R_m; the permittivity of its first layer of dielectric is ϵ_m with thickness $R_{m1} - R_m$; and the permittivity of its second layer is ϵ_{m1} with thickness $R_{m2} - R_{m1}$. On the shield (referred to as the $m = 0$ conductor), the permittivities of the first and second layers are ϵ_0 and ϵ_{01} with thickness $R_0 - R_{01}$ and $R_{01} - R_{02}$, respectively. The permittivity of the material separating the insulated conductors is ϵ.

When the cross-sectional plane of the cable is viewed as the complex plane, the centers of the conductors can be specified as complex numbers. These are denoted b_m ($m = 0, 1, \cdots, M$), where $b_0 = 0$ refers to the center of the shield. A typical cross section is shown in Fig. 1.

With (ρ_m, ϕ_m) as polar coordinates based at the center of the mth conductor ($m = 0, 1, \cdots, M$), a potential function U is assumed to satisfy the following conditions:

(*i*) In the region separating the conductors, $\nabla^2 U = 0$, where $\nabla^2 =$

$(\partial^2/\partial x^2) + (\partial^2/\partial y^2)$ and (x, y) denote Cartesian coordinates in the plane (i.e., U satisfies Laplace's equation).

(*ii*) At the surface of the mth conductor ($\rho_m = R_m$), U has the Fourier series:

$$U = \sum_n u_{nm} \exp(in\phi_m) \qquad m = 0, 1, \cdots, M. \tag{1}$$

(*iii*) When $\tilde{\epsilon}$ denotes the permittivity as a function of position, then both U and $\tilde{\epsilon}(\partial U/\partial \rho_m)$ are continuous across dielectric interfaces about the mth conductor for $m = 0, 1, \cdots, M$. The charge density associated with U at the mth conductor is

$$p_m = \epsilon_m \left.\frac{\partial U}{\partial \rho_m}\right|_{\rho_m = R_m} = \sum_n p_{nm} \exp(in\phi_m) \qquad m = 0, 1, \cdots, M. \tag{2}$$

Thus, the problem is to determine the Fourier components, p_{nm}.

The boundary problem for U is put into the context of complex variables by viewing the cross-sectional plane of the cable as the complex plane. Then, in the region separating the conductors, there is an analytic function $f(z)$ (unique within addition of an imaginary constant) such that U is the real part of $f(z)$. And $f(z)$ can be represented as a Laurent series.[10]

In Appendix A, a Laurent series is calculated for $f(z)$ in a neighborhood of b_m in terms of the Fourier series of U and $\epsilon_m\,(\partial U/\partial \rho_m)$ at the surface of the mth conductor [eq. (1) and eq. (2), respectively] for $m = 0, 1, \cdots, M$. The result, eq. (26) of Appendix A, shows the coefficients of the Laurent series about $z = b_m$ depending only on p_{nm} and u_{nm} ($n = 0, \pm 1, \pm 2, \cdots$) with no explicit indication of interconductor coupling.

A second representation for $f(z)$, which is based on Cauchy's integral formula is

$$f(z) = f_0(z) + \sum_{m=1}^{M} f_m(z), \tag{3}$$

where

$$f_0(z) = \beta_{00} + \sum_{n=1}^{\infty} \beta_{n0} \left(\frac{z}{R_0}\right)^n \tag{4}$$

is analytic everywhere inside the shield and

$$f_m(z) = \beta_{0m} \ln \left(\frac{z - b_m}{R_m}\right) + \sum_{n=1}^{\infty} \beta_{nm} \left(\frac{z - b_m}{R_m}\right)^{-n} \tag{5}$$

is analytic everywhere outside the mth wire for $m = 1, \cdots, M$ (see Ref. 3). The coefficients $\beta_{00}, \cdots, \beta_{0M}$ are real, and in general the β_{nm} (for $1 \leq m \leq M$ and $n > 0$) are complex. In Appendix B, a second Laurent series is calculated for $f(z)$ in a neighborhood of b_m by combining these

functions. The result is eq. (38) for $m = 0$ and eq. (39) for $m = 1, \cdots, M$.

The two Laurent series must be identical since they represent the same function $f(z)$. Therefore, their coefficients can be equated, and, as shown in Appendix C, this leads to systems of linear equations in u_{nm}, p_{nm}, and β_{nm} for $m = 0, 1, \cdots, M$ and $n = 0, 1, \cdots$. It suffices to deal with nonnegative values of n because U and p_m are real quantities for all m (hence, $u_{-nm} = u^*_{nm}$ and $p_{-nm} = p^*_{nm}$, where * denotes complex conjugate).

The systems of equations are conveniently expressed in matrix form with the various coefficients combined into vectors. Accordingly, the following infinite vectors are defined:

$$\begin{aligned} \mathbf{u_m} &= (u_{0m}, u_{1m}, \cdots) \\ \mathbf{p_m} &= (p_{0m}, p_{1m}, \cdots) \\ \boldsymbol{\beta}_\mathbf{m} &= (\beta_{0m}, \beta_{1m}, \cdots), \end{aligned} \tag{6}$$

all for $m = 0, 1, \cdots, M$ and the joint vectors,

$$\begin{aligned} \mathbf{u} &= (\mathbf{u}_0, \cdots, \mathbf{u_m}) \\ \mathbf{p} &= (\mathbf{p}_0, \cdots, \mathbf{p_m}) \\ \boldsymbol{\beta} &= (\boldsymbol{\beta}_0, \cdots, \boldsymbol{\beta}_\mathbf{m}). \end{aligned} \tag{7}$$

As indicated in Appendix C, when these vectors are taken as column vectors, there are matrices T, G, and H such that

$$T\boldsymbol{\beta} = \mathbf{u} \text{ and } \mathbf{p} = G\boldsymbol{\beta} - H\mathbf{u}. \tag{8}$$

In particular, G and H are such that

$$p_{0m} = (\epsilon/R_m)\beta_{0m} \quad m = 1, \cdots, M. \tag{9}$$

If only the zero-order components of the charge-densities are of interest, then it suffices to invert the T-matrix to give

$$\boldsymbol{\beta} = T^{-1}\mathbf{u}, \tag{10}$$

whereupon eq. (9) is used. When all the components of the charge densities are of interest, then combining the equations in (8) gives

$$\mathbf{p} = (GT^{-1} - H)\mathbf{u}. \tag{11}$$

(That the T-matrix is invertible follows from the well-known uniqueness of solutions to Laplace's equation with prescribed values on the boundary.)

In practice, the infinite vectors $\mathbf{u_m}$, $\mathbf{p_m}$, and $\boldsymbol{\beta}_\mathbf{m}$ $m = 0, 1, \cdots, M$ are truncated to give N-vectors, and the matrices T, G, and H are truncated to $(M + 1)N \times (M + 1)N$ matrices. The matrix operations indicated in

eq. (11) are then carried out on the truncated matrices to give an approximation to the first N components of the charge density on each conductor.

A detailed study of the matrix truncation has not been carried out, though the cables studied in Ref. 9 provide some experience in this matter. For the cables whose wires and shield were mutually separated by more than one wire diameter (the 754E and Focal), it more than sufficed to consider five harmonics on the conductors, including the zero order (i.e., $N = 5$). When N was set to 6, there was a difference in the coefficient of the dominant harmonic (zero order) of less than 1 in 10,000; furthermore, the coefficient of the extra harmonic was four orders of magnitude less than that of the zero-order harmonic. For the cable whose wires and shield were separated by a small fraction of one wire diameter (the Proximity cable), eight harmonics were required: the coefficient of the seventh-order harmonic was about $1/15$ that of the zero-order harmonic. When N was set to 10, again there was a difference in the dominant coefficients of less than 1 in 10,000; and the extra coefficients were two orders of magnitude less than that of the zero-order harmonic.

A second practical matter is the presence of conjugation operators in the T-matrix. When it is known (e.g., by symmetry) that the coefficients $\beta_{\ell m}$ are real for all ℓ and m, then the conjugation operator has no effect and can be ignored. The case where the conductor centers are collinear is handled in this way.

In general, the coefficients have an imaginary part. Then $\beta_{\ell m}$ is expressed as the two-vector

$$\begin{pmatrix} \mathrm{Re}\ \beta_{\ell m} \\ \mathrm{Im}\ \beta_{\ell m} \end{pmatrix}$$

and the element of the T-matrix, $T_{km}(n, \ell)$, which multiplies $\beta_{\ell m}$, is expressed as

$$\begin{pmatrix} \mathrm{Re}\ T_{km}(n, \ell) & -\mathrm{Im}\ T_{km}(n, \ell) \\ \mathrm{Im}\ T_{km}(n, \ell) & \mathrm{Re}\ T_{km}(n, \ell) \end{pmatrix}$$

The first component of the matrix product is the proper real part and the second is the proper imaginary part of the product, $T_{km}(n, \ell)\beta_{\ell m}$. When a conjugation operator appears, the 2×2 matrix above must be multiplied on the right by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(this corresponds to changing the sign of the imaginary part of $\beta_{\ell m}$). The result is the matrix

$$\begin{pmatrix} \mathrm{Re}\ T_{km}(n, \ell) & \mathrm{Im}\ T_{km}(n, \ell) \\ \mathrm{Im}\ T_{km}(n, \ell) & -\mathrm{Re}\ T_{km}(n, \ell) \end{pmatrix}.$$

When these matrices are substituted for the elements of the T-matrix, then the resulting matrix can be inverted in the usual way.

III. SUMMARY AND EXTENSIONS

Matrix equations have been derived which relate $\mathbf{u}$, $\mathbf{p}$, and the auxiliary vector β. These are

$$T\beta = \mathbf{u} \text{ and } \mathbf{p} = G\beta - H\mathbf{u}.$$

From these equations, the vector $\mathbf{p}$ can be determined when $\mathbf{u}$ is given. When the potential function U is the constant 1 on the mth conductor and zero on the others, the quantity

$$C_{km} = 2\pi R_k p_{0k} = 2\pi\epsilon\beta_{0k} \quad 1 \leqq k, m \leqq M \tag{12}$$

is the km-element of the $M \times M$ capacitance matrix C. Thus, the capacitance matrix is calculated as a special case of the analysis in Section II.

These results can be readily extended to cables without a shield and with some modification to cables with circular holes in the interstitial dielectric. When no shield is present, two changes must be made:

(*i*) Components of $\mathbf{u}$, $\mathbf{p}$, and β associated with $m = 0$ must be eliminated, and the corresponding submatrices in the matrices T, G, and H must be eliminated.

(*ii*) The T-matrix must be bordered by one row and one column to give

$$\tilde{T} = \begin{pmatrix} 0 & \mathbf{c} \\ \mathbf{c}^t & T \end{pmatrix} \text{ with } \mathbf{c} = (\mathbf{e}_1, \cdots, \mathbf{e}_1), \tag{13}$$

where $\mathbf{e}_1 = (1, 0, 0, \cdots)$ is repeated M-times. Then the first equation in (8) becomes

$$\tilde{T} \begin{pmatrix} \xi \\ \beta \end{pmatrix} = \begin{pmatrix} Q \\ \mathbf{u} \end{pmatrix}, \tag{14}$$

where Q is the total charge of the cable and ξ is a constant to be determined. The second condition comes from the requirement that the total charge be specified, but that the potential on the boundary (the surface of the conductors) be specified only within an additive constant (ξ). This holds for any exterior problem for Laplace's equation in two dimensions.[11]

Circular holes in the interstitial dielectric are treated like extra conductors with unspecified potential values at their surface ($\rho_h = R_h$). But if u_{nh} denote the Fourier components of the potential function at the surface of the hole, then inside the hole,

$$U = \sum_n u_{nh} \left(\frac{\rho_h}{R_h}\right)^n \exp{(\text{i}n\phi_h)}; \tag{15}$$

and if ϵ_h is the permittivity of the material in the hole, then

$$p_h = \epsilon \frac{\partial U}{\partial \rho_h}\bigg|_{\text{out}} = \epsilon_h \frac{\partial U}{\partial \rho_h}\bigg|_{\text{in}} = \sum_n (n\epsilon_h u_{nh}/R_h) \exp(\text{i}n\phi_h), \tag{16}$$

where the normal-derivatives are evaluated on the outside and inside surfaces of the hole as indicated. Therefore, in terms of their Fourier components

$$\mathbf{p}_h = D_h \mathbf{u}_h, \tag{17}$$

where D_h is the diagonal matrix with main diagonal $\{n\epsilon_h/R_h\}$ for $n = 0, 1, 2, \cdots$. Also, by eq. (46)

$$\mathbf{p}_h = G_h \beta_h - H_h \mathbf{u}_h. \tag{18}$$

From eq. (17) and eq. (18), β_h can be calculated in terms of $\mathbf{u}_h$. This in conjunction with the matrix-equation $T\beta = \mathbf{u}$ is sufficient to determine first β_h, then β, and then by the second equation in (8), $\mathbf{p}$. Details are not supplied here.

The problem can also be generalized by specifying more complicated boundary conditions on the conductors. For example, the longitudinal component of the electric field satisfies a boundary condition of the form,

$$\mathbf{p} - S\mathbf{u} = \mathbf{g}, \tag{19}$$

(see Ref. 2) where S is some matrix and $\mathbf{g}$ some vector. It follows immediately from (8) that

$$(G - HT - ST)\beta = \mathbf{g}. \tag{20}$$

Thus, under certain conditions on matrix S, β is obtained by inverting $(G - HT - ST)$, $\mathbf{u}$ is obtained as $T\beta$, and $\mathbf{p}$ is then obtained from eq. (19).

The latter problem is involved in determining the elements of the admittance matrix for the cable (see Ref. 2, eq. 21). As shown in Ref. 2, this is an intermediate step for calculating the propagation modes of the cable and their associated propagation constants. The finite-difference technique, which had been used to calculate charge densities,[7,8] was not capable of dealing with this type of boundary condition. But even for cases such as the calculation of the capacitance matrix where the finite difference technique could be applied, the technique described here has a cost savings of more than one order of magnitude for 0.1 percent accuracy. Thus, the technique has proved to be flexible in solving potential problems associated with uniform cables and relatively inexpensive.

APPENDIX A

The First Laurent Series

When the cross-sectional plane of the cable is viewed as the complex plane, the potential function U is the real part of an analytic function

$f(z)$. In this appendix a Laurent series is deduced for $f(z)$ in a neighborhood of each conductor, starting from the Fourier series for the charge density and U at the surface of the conductor, as given in eq. (1) and eq. (2).

In the first layer of insulation for the mth conductor (for $R_m \leqq \rho_m \leqq R_{m1}$ $m = 1, \cdots, M$ or $R_0 \geqq \rho_0 \geqq R_{01}$), the potential function is

$$U(\rho_m, \phi_m) = u_{0m} + p_{0m}(R_m/\epsilon_m)\ell n\, \xi_m + \tfrac{1}{2} \sum_{n \neq 0}$$

$$\times \exp(in\phi_m)[u_{nm}(\xi_m^n + \xi_m^{-n}) + p_{nm}(R_m/n\epsilon_m)(\xi_m^n - \xi_m^{-n})], \quad (21)$$

where $\xi_m = (\rho_m/R_m)$. This is validated by noting that it satisfies Laplace's equation and it satisfies the boundary conditions of eq. (1) and eq. (2) at $\rho_m = R_m$ (i.e., when $\xi_m = 1$).

In the second layer of insulation (for $R_{m1} \leqq \rho_m \leqq R_{m2}$ $m = 1, \cdots, M$ or $R_{01} \geqq \rho_0 \geqq R_{02}$), the potential function is

$$U(\rho_m, \phi_m) = u_{0m} + p_{0m}(R_m/\epsilon_m)\, \ell n r_m + p_{0m}(R_m/\epsilon_{m1})\ell n \xi_{m1}$$

$$+ \tfrac{1}{4} \sum_{n \neq 0} \exp(in\phi_m)[u_{nm}G_{nm}^{+}(\rho_m) + p_{nm}(R_m/n\epsilon_m)G_{nm}^{-}(\rho_m)], \quad (22)$$

where $\xi_{m1} = (\rho_m/R_{m1})$,

$$r_m = (R_{m1}/R_m), \quad \delta_m = (\epsilon_{m1}/\epsilon_m) \quad (23)$$

and

$$G_{nm}^{\pm}(\rho_m) = (r_m^n \pm r_m^{-n})(\xi_{m1}^n + \xi_{m1}^{-n})$$
$$+ (\delta_m)^{-1}(r_m^n \mp r_m^{-n})(\xi_{m1}^n - \xi_{m1}^{-n}). \quad (24)$$

This satisfies Laplace's equation and the continuity conditions on U and $\tilde{\epsilon}(\partial U/\partial \rho_m)$ at the interface $\rho_m = R_{m1}$ (i.e., when $\xi_{m1} = 1$ and $\xi_m = r_m$). For U this is obvious, but to check the continuity of $\tilde{\epsilon}(\partial U/\partial \rho_m)$, it is useful to refer to the calculation,

$$(n/\rho_m)H_{nm}^{\pm}(\rho_m) \equiv \frac{\partial}{\partial \rho_m} G_{nm}^{\pm}(\rho_m) = (n/\rho_m)[(r_m^n \pm r_m^{-n})(\xi_{m1}^n - \xi_{m1}^{-n})$$
$$+ (\delta_m)^{-1}(r_m^n \mp r_m^{-n})(\xi_{m1}^n + \xi_{m1}^{-n})]. \quad (25)$$

Outside the insulation (for $\rho_m \geqq R_{m2} +, m = 1, \cdots, M$ or $\rho_0 < R_{02}$), the analytic function whose real part matches U is

$$f(z) = u_{0m} + p_{0m}(R_m/\epsilon)\kappa_m + p_{0m}(R_m/\epsilon)\ell n \left(\frac{z - b_m}{R_m}\right)$$

$$+ \tfrac{1}{4} \sum_{n=1}^{\infty} \left(\frac{z - b_m}{R_m}\right)^n \quad [u_{nm}A_{nm} + p_{nm}(R_m/n\epsilon_m)B_{nm}]$$

$$+ \tfrac{1}{4} \sum_{n=1}^{\infty} \left(\frac{z - b_m}{R_m}\right)^{-n} [u_{nm}^{*}E_{nm} + p_{nm}^{*}(R_m/n\epsilon_m)F_{nm}], \quad (26)$$

where * denotes complex conjugation,

$$\kappa_m = \delta_{m1}\delta_m \ell n r_m + \delta_{m1} \ell n r_{m1} - \ell n r_m r_{m1}, \tag{27}$$

$$r_{m1} = (R_{m2}/R_{m1}), \qquad \delta_{m1} = (\epsilon/\epsilon_{m1}), \tag{28}$$

and with $G^{\pm}_{nm} = G^{\pm}_{nm}(R_{m2})$ and $H^{\pm}_{nm} = H^{\pm}_{nm}(R_{m2})$,

$$A_{nm} = (r_m r_{m1})^{-n}(G^{+}_{nm} + (\delta_{m1})^{-1}H^{+}_{nm})$$
$$B_{nm} = (r_m r_{m1})^{-n}(G^{-}_{nm} + (\delta_{m1})^{-1}H^{-}_{nm})$$
$$E_{nm} = (r_m r_{m1})^{n}(G^{+}_{nm} - (\delta_{m1})^{-1}H^{+}_{nm})$$
$$F_{nm} = (r_m r_{m1})^{n}(G^{-}_{nm} - (\delta_{m1})^{-1}H^{-}_{nm}). \tag{29}$$

Substituting $z = b_m + R_{m2} \exp(in\phi_m)$ into $f(z)$ and taking the real part gives

$$u_{0m} + p_{0m}(R_m/\epsilon)(\kappa_m + \ell n r_m r_{m1})$$
$$+ 1/8 \sum_{n \neq 0} \exp(in\phi_m)[u_{nm}(A_{nm}(r_m r_{m1})^n + E_{nm}(r_m r_{m1})^{-n})$$
$$+ p_{nm}(R_m/n\epsilon_m)(B_{nm}(r_m r_{m1})^n + F_{nm}(r_m r_{m1})^{-n})],$$

since $u_{-nm} = u^*_{nm}$ and $p_{-nm} = p^*_{nm}$. As is easily checked, this matches U in eq. (22) for $\rho_m = R_{m2}$ (i.e., for $\xi_{m1} = r_{m1}$). Likewise, it is straightforward to check that the ρ_m-derivative of the real part of $f(z)$ matches $(\epsilon_{m1}/\epsilon_m)\partial U/\partial \rho_m$ evaluated at $\rho_m = R_{m2-}$. This validates eq. (26) for $f(z)$.

When the dielectric is homogeneous, then $r_m = r_{m1} = \delta_m = \delta_{m1} = 1$. It follows that in this case $\kappa_m = 0$ and $A_m = B_{nm} = E_{nm} = -F_{nm} = 4$.

APPENDIX B

The Second Laurent Series

An analytic function $f(z)$ in the region separating the conductors has the form given in eq. (3) through eq. (5). In this appendix, a single Laurent series is derived for $f(z)$ in a neighborhood of each conductor (including the shield) by combining these equations.

The functions $f_m(z)$, as given in eq. (4) and eq. (5), are analytic in a neighborhood of b_k for $m \neq k$ and $k \neq 0$; hence, they are represented by a power series about $z = b_k$,

$$f_m(z) = \sum_{n=0}^{\infty} c_{mk}(n) \left(\frac{z - b_k}{R_k}\right)^n \quad m = 0, 1, \cdots, M. \tag{30}$$

The coefficients $c_{mk}(n)$ are related to the derivatives of $f_m(z)$ at $z = b_k$ by the formula

$$c_{mk}(n) = \frac{(R_k)^n}{n!} \frac{\partial^n f_m}{\partial z^n}\bigg|_{z=b_k} \qquad n = 0, 1, 2, \cdots; \tag{31}$$

in particular, $c_{mk}(0) = f_m(b_k)$.

The results of the calculation in eq. (31) are indicated at the end of this appendix. When the formula is evaluated for $f_0(z)$, as given in eq. (4), the result is eq. (40), and for $f_m(z)$, as given in eq. (5), the result is the combination of eq. (41) and eq. (42). The combined Laurent series for $f(z)$ about $z = b_k$ is indicated in eq. (39).

In a neighborhood of the surrounding shield, the Laurent series of eq. (5) holds for $f_m(z)$ $(m = 1, \cdots, M)$. This is rewritten as

$$f_m(z) = \beta_{0m}\, \ell n \left(\frac{z}{R_0}\right) + \beta_{0m}\, \ell n \left(\frac{R_0}{R_m}\right) + g_m(z), \tag{32}$$

where

$$g_m(z) = \beta_{0m}\, \ell n \left(\frac{z - b_m}{z}\right) + \sum_{\ell=1}^{\infty} \beta_{\ell m} \left(\frac{z - b_m}{R_m}\right)^{-\ell}. \tag{33}$$

Since $g_m(\infty) = 0$, it follows that $g_m(R_0^2/z)$ is analytic in a neighborhood of $z = 0$; so it is represented there by a power series,

$$g_m\left(\frac{R_0^2}{z}\right) = \sum_{n=1}^{\infty} c_{m0}(n) \left(\frac{z}{R_0}\right)^n; \tag{34}$$

and accordingly,

$$g_m(z) = \sum_{n=1}^{\infty} c_{m0}(n) \left(\frac{z}{R_0}\right)^{-n}. \tag{35}$$

The coefficients $c_{m0}(n)$ for $m = 1, \cdots, M$ are obtained from the formula,

$$c_{m0}(n) = \frac{R_0^n}{n!} \frac{d^n}{dz^n}\left[g_m\left(\frac{R_0^2}{z}\right)\right]\bigg|_{z=0} \qquad n = 1, 2, \cdots. \tag{36}$$

The result of applying this formula to eq. (33) is

$$c_{m0}(n) = \left(\frac{b_m}{R_0}\right)^n \left(-\frac{\beta_{0m}}{n} + \sum_{\ell=1}^{n} \beta_{\ell m} \binom{n-1}{\ell-1} \left(\frac{b_m}{R_m}\right)^{-\ell}\right). \tag{37}$$

$$n = 1, 2, \cdots$$

The combined Laurent series for $f(z)$ in a neighborhood of the shield is

$$f(z) = \sum_{n=0}^{\infty} \beta_{n0} \left(\frac{z}{R_0}\right)^n + \sum_{m=1}^{M} \left[\beta_{0m}\, \ell n \left(\frac{z}{R_0}\right) + \beta_{0m}\, \ell n \left(\frac{R_0}{R_m}\right) + \sum_{n=1}^{\infty} c_{m0}(n) \left(\frac{z}{R_0}\right)^{-n}\right]. \tag{38}$$

The combined Laurent series for $f(z)$ about $z = b_k$ $(k \neq 0)$ is

$$f(z) = \sum_{m \neq k} c_{mk}(0) + \beta_{0k}\, \ell n \left(\frac{z - b_k}{R_k}\right) + \sum_{n=1}^{\infty} \beta_{nk} \left(\frac{z - b_k}{R_k}\right)^{-n} + \sum_{n=1}^{\infty} \left(\frac{z - b_k}{R_k}\right)^{n} \left(\sum_{m \neq k} c_{mk}(n)\right). \tag{39}$$

The coefficients obtained by carrying out the calculations in eq. (31) are

$$c_{0k}(n) = \left(\frac{b_k}{R_k}\right)^{-n} \sum_{\ell=n}^{\infty} \beta_{\ell 0} \binom{\ell}{n} \left(\frac{b_k}{R_0}\right)^{\ell} \quad k \neq 0, n = 0, 1, 2, \cdots \tag{40}$$

and for $m, k \neq 0$, $m \neq k$

$$c_{mk}(0) = \beta_{0m}\, \ell n \left|\frac{b_k - b_m}{R_m}\right| + \sum_{\ell=1}^{\infty} \beta_{\ell m} \left(\frac{b_k - b_m}{R_m}\right)^{-\ell} \tag{41}$$

$$c_{mk}(n) = \left(\frac{b_m - b_k}{R_k}\right)^{-n} \times \left[-\frac{\beta_{0m}}{n} + \sum_{\ell=1}^{\infty} \beta_{\ell m} \binom{\ell + n - 1}{n} \left(\frac{b_k - b_m}{R_m}\right)^{-\ell}\right] \tag{42}$$

for $n = 1, 2, \cdots$.

APPENDIX C

Equate the Two Laurent Series

Equating the constant, logarithmic, nth power, and $-n$th power terms $(n \geqq 1)$, respectively, in the Laurent series about the shield [i.e., in eq. (26) and eq. (38)] gives the linear equations,

$$\begin{aligned}
&(i)\ \beta_{00} + \sum_{m=1}^{M} \beta_{m0}\, \ell n \frac{R_0}{R_m} = u_{00} + p_{00}(R_0/\epsilon)\kappa_0 \\
&(ii)\ \sum_{m=1}^{M} \beta_{0m} = p_{00}(R_0/\epsilon) \\
&(iii)\ \beta_{n0} = \tfrac{1}{4}[u_{n0}A_{n0} + p_{n0}(R_0/n\epsilon_0)B_{n0}] \\
&(iv)\ \sum_{m=1}^{M} c_{m0}(n) = \tfrac{1}{4}[u^*_{n0}E_{n0} + p^*_{n0}(R_0/n\epsilon_0)F_{n0}].
\end{aligned} \tag{43}$$

For the kth wire $(k = 1, \cdots, M)$, the corresponding equations are

$$\begin{aligned}
&(i)\ \sum_{m \neq k} c_{mk}(0) = u_{0k} + p_{0k}(R_k/\epsilon)\kappa_k \\
&(ii)\ \beta_{0k} = p_{0k}(R_k/\epsilon) \\
&(iii)\ \sum_{m \neq k} c_{mk}(n) = \tfrac{1}{4}[u_{nk}A_{nk} + p_{nk}(R_k/n\epsilon_k)B_{nk}] \\
&(iv)\ \beta_{nk} = \tfrac{1}{4}[u^*_{nk}E_{nk} + p^*_{nk}(R_k/n\epsilon_k)F_{nk}].
\end{aligned} \tag{44}$$

The charge coefficients can be solved from (*ii*) and (*iii*) in the first set and from (*ii*) and (*iv*) in the second set to give

$$
\begin{aligned}
p_{00} &= (\epsilon/R_0) \sum_{m=1}^{M} \beta_{0m} \\
p_{n0} &= (4\epsilon_0 n/R_0 B_{n0})\beta_{n0} - (\epsilon_0 n A_{n0}/R_0 B_{n0}) u_{n0} \\
p_{0k} &= (\epsilon/R_k)\beta_{0k} \\
p_{nk} &= (4\epsilon_k n/R_k F_{nk})\beta^{*}_{nk} - (\epsilon_k n E_{nk}/R_k F_{nk}) u_{nk}.
\end{aligned} \tag{45}
$$

In terms of the vectors defined in (6) (viewed as column vectors), these equations are expressed

$$
\mathbf{p}_k = G_k \boldsymbol{\beta}_k - H_k \mathbf{u}_k \qquad k = 0, 1, \cdots, M, \tag{46}
$$

where the $n\ell$-elements of these matrices for $n, \ell = 0, 1, \cdots$ are

$$
G_0(n, \ell) = \begin{cases} (0; \mathbf{e}_1; \cdots; \mathbf{e}_1) & n = 0 \\ (4\epsilon_0 n/R_0 B_{n0}) & n = \ell \neq 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\begin{array}{c} G_k(n, \ell) = \\ (k = 1, \cdots, M) \end{array} \begin{cases} (\epsilon/R_k) & n = \ell = 0 \\ (4\epsilon_k n/R_k F_{nk})(*) & n = \ell \neq 0 \\ 0 & n \neq \ell \end{cases}
$$

$$
H_0(n, \ell) = \begin{cases} (\epsilon_0 n A_{n0}/R_0 B_{n0}) & n = \ell \\ 0 & n \neq \ell \end{cases}
$$

$$
\begin{array}{c} H_k(n, \ell) = \\ (k = 1, \cdots, M) \end{array} \begin{cases} (\epsilon_k n E_{nk}/R_k F_{nk}) & n = \ell \\ 0 & n \neq \ell \end{cases}
$$

In terms of the joint vectors defined in (7),

$$
\mathbf{p} = G\boldsymbol{\beta} - H\mathbf{u}, \tag{47}
$$

where G and H are the direct sum of the matrices, G_k and H_k, respectively, for $k = 0, 1, \cdots, M$ (see Ref. 12, p. 159).

The charge components can be eliminated in the original set of equations by means of the identity,

$$
\begin{gathered}
B_{nk}E_{nk} - A_{nk}F_{nk} = (32/\delta_k \delta_{k1}) \\
n = 0, 1, \cdots \quad \text{and} \quad k = 0, \cdots, M.
\end{gathered} \tag{48}
$$

The result is the set of equations,

$$(i)\ \beta_{00}+\sum_{m=1}^{M}\beta_{0m}\left(\ell n\frac{R_0}{R_m}-\kappa_0\right)=u_{00}$$

$$(ii)\ -(\delta_0\delta_{01}F_{n0}/8)\beta_{n0}+(\delta_0\delta_{01}B_{n0}/8)\sum_{m=1}^{M}c^*_{m0}(n)=u_{n0}$$

$$(iii)\ \sum_{m\neq k}c_{mk}(0)-\beta_{0k}\kappa_k=u_{0k}$$

$$(iv)\ -(\delta_k\delta_{k1}F_{nk}/8)\sum_{m\neq k}c_{mk}(n)+(\delta_k\delta_{k1}B_{nk}/8)\beta^*_{nk}=u_{nk}$$

for $k=1,\cdots,M$ and $n=1,2,\cdots$. (49)

These equations can be expressed in the matrix form

$$T\beta\equiv\begin{bmatrix}T_{00}T_{01}\cdots T_{0M}\\ T_{10}T_{11}\cdots T_{1M}\\ \vdots\\ T_{M0}T_{M1}\cdots T_{MM}\end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_M\end{bmatrix}=\begin{bmatrix}\mathbf{u}_0\\ \mathbf{u}_1\\ \vdots\\ \mathbf{u}_M\end{bmatrix}\equiv\mathbf{u},\tag{50}$$

where β and $\mathbf{u}$ are viewed as column vectors. The symbols T_{km} for $k, m = 0, \cdots, M$ denote submatrices of T defined as follows:

$$T_{00}(n,\ell)=\begin{cases}1 & n=\ell=0\\ (\delta_0\delta_{01}F_{n0}/8) & n=\ell\neq 0\\ 0 & n\neq\ell\end{cases}$$

$$\begin{array}{l}T_{0m}(n,\ell)=\\ (m=1,\cdots,M)\end{array}\begin{cases}\ell n\dfrac{R_0}{R_m}-\kappa_0 & n=\ell=0\\ -(\delta_0\delta_{01}B_{n0}/8n)\left(\dfrac{b_m}{R_0}\right)^n\ (*) & \ell=0;\ n=1,2,\cdots\\ (\delta_0\delta_{01}B_{n0}/8)\left(\dfrac{b_m}{R_0}\right)^n\left(\dfrac{b_m}{R_m}\right)^{-\ell}\dbinom{n-1}{\ell-1} & \ell=1,\cdots,n;\ n=1,2,\cdots\\ 0 & \text{otherwise}\end{cases}$$

$$\begin{array}{l}T_{k0}(n,\ell)=\\ (k=1,\cdots,M)\end{array}\begin{cases}\left(\dfrac{b_k}{R_0}\right)^\ell & n=0;\ \ell=0,1,\cdots\\ (\delta_k\delta_{k1}F_{nk}/8)\dbinom{\ell}{n}\left(\dfrac{b_k}{R_k}\right)^{-n}\left(\dfrac{b_k}{R_0}\right)^\ell & 1\leqq n\leqq\ell;\ \ell=1,2,\cdots\\ 0 & \text{otherwise}\end{cases}$$

$$
\begin{aligned}
&T_{kk}(n, \ell) = \\
&(k = 1, \cdots, M)
\end{aligned}
\begin{cases}
-\kappa_k & n = \ell = 0 \\
(\delta_{k1}\delta_k B_{nk}/8)(*) & n = \ell \neq 0 \\
0 & n \neq \ell
\end{cases}
$$

$$
\begin{aligned}
T_{km}(n, \ell) = \quad & \ell n \left| \frac{b_k - b_m}{R_m} \right| && n = \ell = 0 \\
(k, m \neq 0 \quad k \neq m) \quad & \left(\frac{b_k - b_m}{R_m} \right)^{-\ell} && n = 0; \ell = 1, 2, \cdots \\
& (\delta_k \delta_{k1} F_{nk}/8n) \left(\frac{b_m - b_k}{R_k} \right)^{-n} && \ell = 0; \; n = 1, 2, \cdots \\
& -(\delta_k \delta_{k1} F_{nk}/8) \binom{\ell + n - 1}{n} \left(\frac{b_m - b_k}{R_k} \right)^{-n} \left(\frac{b_k - b_m}{R_m} \right)^{-\ell} && \\
& && n, \ell \neq 0.
\end{aligned}
$$

The symbol (*) throughout means that complex conjugation is to be performed. These elements of the various submatrices are read off the appropriate coefficients in the system of equations in (49) in conjunction with the equations for the $c_{mk}(\cdot)$ in eqs. (37), (40), (41), and (42).

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